Let $a_1, a_2, \ldots a_n, (n \geq 3)$ be distinct complex numbers. Compute the sum

$$\sum_{k=1}^{n} s_k \prod_{j \neq k} \frac{(-1)^n}{a_j - a_k},$$

where $s_k = \left(\sum_{i=1}^{n} a_i\right) - a_k,\ 1 \leq k \leq n.$

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\ dx}.$$

Solutions

- **5224:** *Proposed by Kenneth Korbin, New York, NY*

  Let $T_1 = T_2 = 1, T_3 = 2$, and $\mathrm{T}_N = T_{N-1} + T_{N-2} + T_{N-3}$. Find the value of

$$\sum_{N=1}^{\infty} \frac{T_N}{\pi^N}.$$

Solution 1 by Arkady Alt, San Jose, CA

Noting that $\{T_n\}_{n\geq 1}$ is an increasing sequence of positive integers we obtain:

$$\begin{aligned}
\frac{T_{n+1}}{T_n} &= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_n} \\
&= 1 + \frac{T_{n-1}}{T_n} + \frac{T_{n-2}}{T_{n-1}} \cdot \frac{T_{n-1}}{T_n} \\
&< 1 + 1 + 1 \cdot 1 = 3,\ n \in N.
\end{aligned}$$

Hence,

$$\frac{T_{n+1}}{T_n} < 3 \iff \frac{T_{n+1}}{3^{n+1}} < \frac{T_n}{3^n},\ n \in N \implies \frac{T_n}{3^n} < \frac{T_1}{3^1} \iff T_n < 3^{n-1},\ n \in N.$$

and therefore, by the comparison test for series, $\sum_{i=1}^{n} T_i x^{i-1}$ is convergent for any $x \in \left(0, \frac{1}{3}\right)$ because for such x it is bounded by $\sum_{n=1}^{\infty} (3x)^{n-1} = \frac{1}{1-3x}$.

Since

$$\left(1 - x - x^2 - x^3\right) \sum_{n=1}^{\infty} T_n x^{n-1} = T_1 + x(T_2 - T_1) + x^2(T_3 - T_2 - T_1)$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} x^{n+2}\left(T_{n+3}-T_{n+2}-T_{n+2}-T_{n}\right) \\
&= T_1 + x(1-1) + x^2(2-1-1) + \sum_{n=1}^{\infty} x^{n+2}\cdot 0 = 1
\end{aligned}
$$

then
$$\sum_{n=1}^{\infty} T_n x^{n-1} \frac{1}{1-x-x^2-x^3} \iff \sum_{n=1}^{\infty} T_n x^n = \frac{x}{1-x-x^2-x^3}$$
and therefore, for $x = \dfrac{1}{\pi} < 3$, we obtain
$$\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{\frac{1}{\pi}}{1-\frac{1}{\pi}-\frac{1}{\pi^2}-\frac{1}{\pi^3}} = \frac{\pi^2}{\pi^3-\pi^2-\pi-1}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

We first claim that $1 \le T_n \le 2^{n-1}$ for $n \ge 1$. Indeed this is true for $n = 1, 2$, and 3 and

$1 \le T_n = T_{n-1}+T_{n-2}+T_{n-3} \le 2^{n-2}+2^{n-3}+2^{n-4} < 2^{n-2}+2^{n-3}+2^{n-3} = 2^{n-1}$, as claimed.

So, $S = \displaystyle\sum_{n=1}^{\infty} \frac{T_n}{\pi^n}$ is convergent and

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{T_n}{\pi^n} = \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \sum_{n=1}^{\infty} \frac{T_{n-1}+T_{n-2}+T_{n-3}}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\sum_{n=3}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^2}\sum_{n=2}^{\infty} \frac{T_n}{\pi^n} + \frac{1}{\pi^3}\sum_{n=1}^{\infty} \frac{T_n}{\pi^n} \\
&= \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{2}{\pi^3} + \frac{1}{\pi}\left(S - \frac{1}{\pi} - \frac{1}{\pi^2}\right) + \frac{1}{\pi^2}\left(S - \frac{1}{\pi}\right) + \frac{1}{\pi^3} S \\
&= \frac{1}{\pi} + S\left(\frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3}\right). \quad \text{So,} \\
S &= \frac{\pi^2}{\pi^3-\pi^2-\pi-1}
\end{aligned}
$$

Solution 3 by Adrian Naco, Polytechnic University, Tirana, Albania

Let us pose, $a_n = \dfrac{T_n}{\pi^n}$, $T_0 = 0$. We prove by induction that, $T_n \le T_{n+1} \le 2T_n$.

$$T_n \le T_{n+1} = T_n + T_{n-1} + T_{n-2} \le 2T_{n-1} + 2T_{n-2} + 2T_{n-3} = 2T_n.$$

Thus, it implies that,

$$\forall n \in N: \qquad \frac{1}{\pi}a_n \le a_{n+1} = \frac{T_{n+1}}{\pi^{n+1}} = \frac{1}{\pi}\cdot\frac{T_{n+1}}{T_n}\cdot\frac{T_n}{\pi^n} \le \frac{2}{\pi}a_n,$$